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Transmission of an acoustic pulse through a plane vortex sheet

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This paper discusses the linear theory of the transmission of an acoustic pulse through a plane discontinuity of velocity. It is shown that elementary ideas of geometrical acoustics which have received much attention in the recent literature lead to the erroneous prediction of a zone of silence. It is in precisely this zone that unstable disturbances and broad-fronted pulses of enhanced intensity propagate, having been triggered-off by the arrival of the pulse at the vortex sheet. The apparent qualitative agreement between geometrical acoustics and experimental data regarding sound radiation from the interior of supersonic jets is shown to be purely fortuitous, and it is argued that a complete analysis of such problems must depend on a deeper and possibly non-linear treatment.

1. Introduction

The study of acoustic radiation from the interior of supersonic jets has attracted much attention in recent years (Ribner 1964; Berman & Ffowcs Williams 1970). It is generally recognized that refraction at the surface of the jet alters the nature of the radiation field, but the extent to which this modifies the simple picture of acoustic radiation from convected turbulent quadrupoles (Lighthill 1952) is still an open question. Ribner (1964) suggests that such effects are confined merely to the creation of a zone of silence downstream of the acoustic sources, so that the sound is refracted 'outwards' from the jet. However, a large body of work on the problem of jet stability, culminating in the comprehensive study of Miles (1958), leads one to suspect that any attempt to extrapolate classical notions of geometrical acoustics may result in a misleading, or indeed grossly inaccurate, picture of the radiation field. A more careful analysis which takes account of possible instabilities is now required.

The present study arose from an attempt to apply such ideas to the calculation of the radiation field due to a *stationary* source within a supersonic jet. In particular such a source would model the acoustic collision product of a turbulent pocket with one of the compression waves which occur at roughly equal intervals along an imperfectly expanded supersonic jet of finite diameter. These waves focus onto the wall of the jet causing the sharp necking characteristic of the cellular structure of such jets. Since sound radiation within the jet can only pass downstream of such a source, it was quickly realized that the exact nature of the geometry of the jet in the vicinity of the source is largely irrelevant, the problem reducing to that of calculating the field transmitted through a velocity discontinuity. Further, the motion being fully supersonic, the precise form of the source is unimportant, and we shall present an idealized problem in which the far field of the acoustic radiation transmitted through a *plane vortex sheet* is examined. The source of sound will be assumed to lie close to the velocity discontinuity.

In modelling the jet in this way we are neglecting effects due to finite jet diameter. The most important limitation on our analysis results from the fact that natural oscillations of the vortex sheet, which in our case are neutrally stable, can definitely be shown to be unstable for a finite diameter jet (Berman & Ffowcs Williams 1970). Such instability becomes significant when the acoustic coupling between opposite faces of the jet becomes large, and occurs after a time which is long enough for sound to bounce backwards and forwards several times between these faces. Our analysis essentially calculates the transmitted field due to that sound which is emitted before these finite diameter effects become important.

The problem of the transmission and reflexion of sound at a plane velocity discontinuity was considered briefly by Rayleigh (1945), who, however, subsequently applied his results incorrectly to the determination of the path of an acoustic ray through a fluid of continuously varying velocity. The first correct treatments of the problem of the transmission of a *plane* acoustic wave through an interface of relative motion appear to be due to Miles (1957) and Ribner (1957). Earlier attempts by Rudnick (1946), Keller (1955) and Franken & Ingard (1956), are shown by Miles to be in error. All three papers apply the kinematic boundary condition at the interface incorrectly, and in addition the last two papers fail to apply the appropriate radiation condition.

In two later papers (Miles 1958; Miles & Fejer 1963) the stability of the velocity discontinuity (vortex sheet) is studied, and in particular an asymptotic approximation to the displacement of the vortex sheet following a suddenly imposed, spatially periodic velocity is obtained. These papers essentially clarify and correct the earlier work of Landau (1944), Hatanaka (1949) and Pai (1954). Each of these authors ignored the existence of branch points for the eigenvalue equation and accepted the eigenvalues given by its two possible branches. Miles confirms their results regarding stability, but rules out certain of their neutral eigenvalues, or *resonances*.

An extention of this earlier work was attempted by Gottlieb (1960) who considered the problem of a source near a velocity discontinuity. He treats in detail a line source oscillating at a fixed frequency, and sketches the extention of his results to the case of an oscillating point source. Since Mach numbers no greater than about 1·2 are considered, he gives no discussion of the resonances predicted by Miles when the Mach number exceeds 2. Indeed apart from a slight peaking, his graphical results do not predict substantially greater wave amplitudes than if the source were radiating into a homogeneous medium. More importantly from our point of view, he dismisses completely the unstable modes due to Helmholtz instability of the interface, which in any steady state problem would give an infinitely large contribution. Gottlieb assumes that such instabilities are the result of large wavelength disturbances of the order of the jet diameter, and therefore claims to confine himself to the case of disturbances which emit sound of wavelength much smaller than the diameter. Actually consideration of a simple 'switch-on' mechanism for his source, the switch-on time subsequently being allowed to tend to minus infinity, reveals that in deriving his solution Gottlieb has omitted a term corresponding to the crossing of a pole in the complex frequency plane, and for Mach numbers less than $2\sqrt{2}$, this would always give an infinitely large contribution.

It is therefore not possible to regard Gottlieb's results as forming a basis for the synthesis by Fourier analysis of the solution for more general source distributions. Further, it will be seen that when resonances are possible they actually *suppress* his geometrical acoustics solution, and result in the broadening out of the transmitted signal in the resonant directions together with a large increase in intensity. Evidently the harmonic problem treated by Gottlieb is irrelevant in so far as it is desired to give a proper interpretation of an essentially unstable situation.

In the more recent discussion of this problem given in the treatise of Morse & Ingard (1968) the authors completely ignore the existence of instabilities and resonances. Their treatment is purely one of geometrical acoustics which is claimed to predict a definite shadow zone downstream of the source, in spite of the fact that it is in precisely this region that exponentially growing disturbances develop!

In order to obtain a clear grasp of the physical processes involved in the problem of acoustic radiation from a jet, it would appear to be appropriate, following Miles (1958), to formulate an initial value problem. Ideally one would like to observe the response of an initially steady, undisturbed flow to an impulse, which corresponds to determining the Green's function of the problem. To be sure the inevitable final state is one of exponential growth, but now we have the immediate interpretation of the solution as one in which much more sound is emitted by the system than was initially supplied by the source, and further, that the energy of this excess radiation, which tends to be confined to the region downstream of the source, is extracted directly from the shear layer.

This approach has been developed in a recent paper by Friedland & Pierce (1969), who consider the transient solution for the *reflexion* of line source generated waves by a plane vortex sheet. In their problem, however, the source is at rest relative to the fluid of the jet and, although they are able to derive an exact solution for the reflected wave, the physically more significant transmitted wave is set aside as being too difficult.

The analysis of the present paper is essentially similar to that of Friedland & Pierce except that in the three-dimensional problem treated here exact evaluation of the Fourier integrals involved is not possible and recourse must be had to asymptotic methods. However, it is suggested that our results will be of more value in providing further insight into the problem of noise radiation from supersonic jet aircraft, since the sound transmitted from the interior of the jet is of primary concern in such applications.

Previous work based on linear theory (Ribner 1964) has resulted in the idea, mentioned above, that the existance of a zone of silence causes the sound to be refracted outwards from the jet. In fact experiments indicate that this conclusion is qualitatively correct, but we shall see that an explanation based on linear theory is definitely wrong. A correct linear analysis leads to conclusions which cannot be predicted by a simple ray theory. Presumably, therefore, a proper interpretation of the experimental results must await a deeper study which takes account of non-linear effects.

2. Formulation of the problem

Consider two ideal fluids occupying the half-spaces $z < 0$ and $z > 0$, and characterized by subscripts 1 and 2 respectively. Without losing any significant features of the problem, suppose that both fluids have the same sound speed, a, and are of equal density. The fluid in the lower half space is assumed to have a

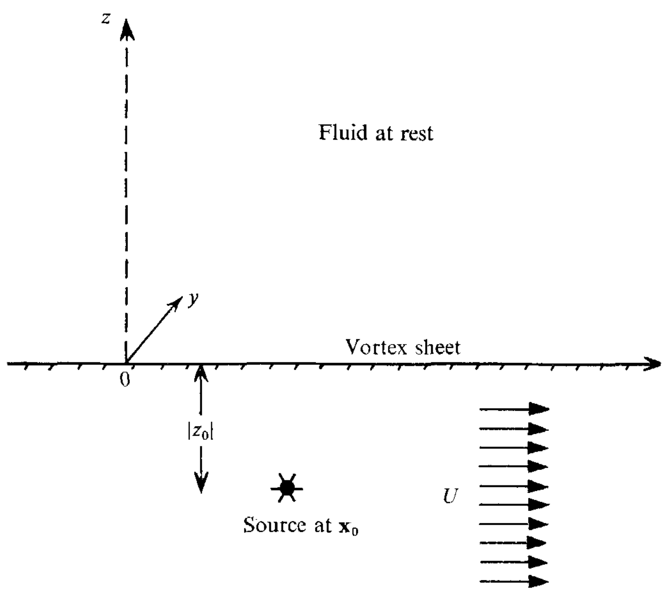

FIGURE 1. The initial configuration of the vortex sheet and point source.

uniform velocity U parallel to the x axis, and, with the y axis chosen such that xyz form a left-handed triad, to have a source $\delta(\mathbf{x}-\mathbf{x}_0)\,\delta(t-t_0)$ at the point $\mathbf{x}_0$ ($z_0 < 0$) near the interface (figure 1). The fluid in $z > 0$ is assumed to be at rest.

Let ϕ_1 and ϕ_2 denote the velocity potentials of the small disturbed motions in the two regions. Then

$$\left.\begin{aligned}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2\phi_1-a^2\nabla^2\phi_1 &= \delta(\mathbf{x}-\mathbf{x}_0)\,\delta(t-t_0)\\ \frac{\partial^2\phi_2}{\partial t^2}-a^2\nabla^2\phi_2 &= 0\end{aligned}\right\} \quad (2.1)$$

in $z < 0$, and

in $z > 0$.

The boundary conditions relating ϕ_1 and ϕ_2 on the interface follow from requiring (i) continuity of pressure, (ii) continuity of normal velocity of the vortex sheet. These are readily applied when the velocity field is decomposed into an assembly of plane wave contributions. The problem then is to determine ϕ_1 and ϕ_2 satisfying conditions (i) and (ii); together they constitute the Green's function of the problem.

Following Miles (1957) and Gottlieb (1960) it is convenient to start by writing down the solution to the first of equations (2.1) in free space, and then to introduce reflected and transmitted waves to account for the presence of the vortex sheet. The boundary conditions (i) and (ii) then provide sufficient information to determine these additional waves. Plane wave decomposition of a velocity field $\phi(\mathbf{x}, t)$, say, is achieved by defining its Fourier transform $\psi(\mathbf{k}, \omega)$ by

$$\left.\begin{aligned} \psi(\mathbf{k},\omega) &= \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d\mathbf{x}\int_{-\infty}^{\infty} \phi(\mathbf{x},t)\, e^{-i(\mathbf{k}.\mathbf{x}-\omega t)}\, dt, \\ \phi(\mathbf{x},t) &= \int_{-\infty}^{\infty} d\mathbf{k}\int_{-\infty+i\epsilon}^{\infty+i\epsilon} \psi(\mathbf{k},\omega)\, e^{i(\mathbf{k}.\mathbf{x}-\omega t)}\, d\omega, \end{aligned}\right\} \tag{2.2}$$

where $\epsilon > 0$.

For a function $\phi(\mathbf{x}, t)$, vanishing for $t < t_0$ and representing a stable disturbance, $\psi(\mathbf{k}, \omega)$, for each fixed wave-number vector $\mathbf{k}$, is a regular function of the frequency ω throughout the upper complex ω plane. On the other hand, if instabilities are present which grow exponentially with time, $\psi(\mathbf{k}, \omega)$ will possess singularities in the upper half-plane, and then to ensure that the solution vanishes for $t < t_0$ the positive quantity ϵ must be large enough for the path of integration in the ω plane to pass above all these singularities. Lighthill (1960) has given a comprehensive discussion of these points.

3. The transmitted pulse. Geometrical acoustics contribution

The procedure outlined above is essentially the same as that given by Gottlieb (1960) and followed by Friedland & Pierce (1969), and we shall be content merely to write down the Fourier integral representation of the solution. We are concerned here only with the transmitted wave ϕ_t, say, and it is an easy matter to show that

$$\phi_t = \frac{i}{(2\pi)^3 a^2}\iint_{-\infty}^{\infty} dl\, dm \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{d\omega\, \omega(\omega-Ul)\exp i\{l(x-x_0)+m(y-y_0) +\gamma_1 z-\gamma_2 z_0-\omega(t-t_0)\}}{[\omega^2\gamma_2+(\omega-Ul)^2\gamma_1]} \tag{3.1}$$

($z \geqslant 0$), where $\epsilon > 0$ is large enough for the integration path in the ω plane to lie above all the singularities of the integrand, i.e. of the transmission coefficient. The functions γ_1 and γ_2 are defined by

$$\left.\begin{aligned} \gamma_1 &= \left(\frac{\omega^2}{a^2}-(l^2+m^2)\right)^{\frac{1}{2}}, \\ \gamma_2 &= \left(\frac{(\omega-Ul)^2}{a^2}-(l^2+m^2)\right)^{\frac{1}{2}}, \end{aligned}\right\} \tag{3.2}$$

each having a positive imaginary part along the line of integration in the ω plane, and l and m are respectively the wave-number components in the x and y directions.

It may be verified that the causality condition is satisfied by the solution (3.1), i.e. that it vanishes for $t < t_0 - z_0/a$, at which time the pulse first impinges on the vortex sheet. When this inequality is not satisfied the integral may be evaluated for $a(t-t_0) > z - z_0$ by collapsing the line of integration about a closed contour in the ω plane on which the phase is real. This is the procedure employed by Friedland & Pierce and takes account of any poles of the integrand which are crossed over, the branch lines joining the branch points of γ_1 and of γ_2 on the real axis being completely contained within the contour. In their case the final contour was an ellipse together with a contribution about the γ_2 branch line. Our case is similar, but only reduces to an ellipse, plus that portion of the γ_2 branch line lying outside the ellipse, when $|z_0| \ll z$. In this case the ellipse is given by

$$\frac{\xi^2}{a^2(t-t_0)^2} + \frac{\eta^2}{z^2} = \frac{a^2(l^2+m^2)}{a^2(t-t_0)^2 - z^2}, \tag{3.3}$$

where $\omega = \xi + i\eta$. Any pole of the integrand which lies inside the contour gives no contribution to the solution. The details are similar to those of Friedland & Pierce and will not be repeated here.

The contribution from the integration about the contour cannot be evaluated exactly and the method of stationary phase may be used to obtain the *geometrical acoustics* contribution to the transmitted wave. In presenting these results it is convenient to divide the far field into three regions defined by the following inequalities.

Region I

$$1 - \frac{M(x-x_0)}{|\mathbf{x}|} > \frac{[(x-x_0)^2 + (y-y_0)^2]^{\frac{1}{2}}}{|\mathbf{x}|}.$$

Here $M = U/a$ is the Mach number of the jet flow, and

$$|\mathbf{x}| = [(x-x_0)^2 + (y-y_0)^2 + z^2]^{\frac{1}{2}}.$$

The classical refracted wave occupies this region and was treated by Gottlieb (1960).

Region II

$$\left|1 - \frac{M(x-x_0)}{|\mathbf{x}|}\right| < \frac{[(x-x_0)^2 + (y-y_0)^2]^{\frac{1}{2}}}{|\mathbf{x}|}.$$

This is the *zone of relative silence* and contains the so called *refracted arrival wave* (cf. Friedland & Pierce 1969, §V).

Region III

$$1 - \frac{M(x-x_0)}{|\mathbf{x}|} < -\frac{[(x-x_0)^2 + (y-y_0)^2]^{\frac{1}{2}}}{|\mathbf{x}|}.$$

This region was absent from Gottlieb's analysis since it exists only for $M > 2$. It is here that the resonant modes propagate, and may be compared with the corresponding region of Friedland & Pierce.

Let us now define the quantities $T(\mathbf{x})$, the transmission coefficient for waves arriving at the point $\mathbf{x}$ $(z > 0)$; and $\gamma(\mathbf{x})$, which is proportional to the projection of the unit normal of these waves onto the z axis prior to their refraction at the jet interface

$$T(\mathbf{x}) = \frac{\dfrac{2z}{|\mathbf{x}|}\left(1-\dfrac{M(x-x_0)}{|\mathbf{x}|}\right)}{\dfrac{z}{|\mathbf{x}|}\left(1-\dfrac{M(x-x_0)}{|\mathbf{x}|}\right)^2 \pm \left[\left(1-\dfrac{M(x-x_0)}{|\mathbf{x}|}\right)^2 - \dfrac{[(x-x_0)^2+(y-y_0)^2]}{|\mathbf{x}|}\right]^{\frac{1}{2}}}, \tag{3.4}$$

$$\gamma(\mathbf{x}) = \pm\left|\left(1-\frac{M(x-x_0)}{|\mathbf{x}|}\right)^2 - \frac{[(x-x_0)^2+(y-y_0)^2]}{|\mathbf{x}|^2}\right|^{\frac{1}{2}}. \tag{3.5}$$

In these expressions the positive sign is taken when $\mathbf{x}$ lies in region I and the negative sign in region III. The zone of relative silence, region II, will be considered separately.

The elementary stationary phase analysis gives the following expression for the geometrical acoustic field, ϕ_A, in regions I and III:

$$\phi_A = \frac{1}{4\pi a^2}\frac{T(\mathbf{x})}{|\mathbf{x}|}\,\delta\!\left(t-t_0-\frac{|\mathbf{x}|}{a}+\frac{z_0\gamma}{a}\right). \tag{3.6}$$

The argument of the δ function is correct to terms of order $(z_0/z)^2$, its vanishing having the obvious interpretation as representing the arrival of the refracted pulse at $\mathbf{x}$ at time t, the pulse having travelled by the least time path from the source determined by the stationary phase calculation. For equal values of $|\mathbf{x}|$ the delay time is seen to be greater in the upstream region I than in region III (see figure 2); this is because while the pulse is still in the jet it is convected downstream at speed U so that waves which eventually radiate into region I are retarded and those into region III speeded-up. Note also that as $M \to 0$ region II and III cease to exist, and $T \to 1$, and the field of a source radiating into a homogeneous medium is recovered.

The form of the field in the zone of relative silence is somewhat different. The pulse still has a sharp front, but now there is no singularity, the field switching on to a finite level at time

$$t_0+\frac{|\mathbf{x}|}{a}-\frac{z_0}{a}\left|\left(1-\frac{M(x-x_0)}{|\mathbf{x}|}\right)^2-\frac{[(x-x_0)^2+(y-y_0)^2]}{|\mathbf{x}|^2}\right|^{\frac{1}{2}}.$$

As noted by Gottlieb, the least-time path for reaching an observer in the zone of relative silence requires attenuation of the wave as it travels from the source to the surface of the jet. This is reflected in the solution (3.1) by the appearance of the factor $\exp(-i\gamma_2 z_0)$, the argument of which is real and negative for waves propagating into region II. The formal expression for the acoustic mode in region II has the form

$$\phi_A = \frac{\tau(\mathbf{x})}{4\pi a^2|\mathbf{x}|}\frac{\left[\dfrac{|\mathbf{x}|}{a}-(t-t_0)-\dfrac{z_0 z}{a|\mathbf{x}|}\left(1-\dfrac{M(x-x_0)}{|\mathbf{x}|}\right)^2\right]H\left(t-t_0-\dfrac{|\mathbf{x}|}{a}+\dfrac{z_0\sigma}{a}\right)}{\left(\dfrac{|\mathbf{x}|}{a}-(t-t_0)\right)^2+\dfrac{z_0^2}{a^2}\sigma^2}, \tag{3.7}$$

where H is the Heaviside unit function and

$$\sigma = \left| \left(1 - \frac{M(x-x_0)}{|\mathbf{x}|}\right)^2 - \frac{[(x-x_0)^2+(y-y_0)^2]}{|\mathbf{x}|^2} \right|^{\frac{1}{2}},$$

$$\tau(\mathbf{x}) = \frac{2z}{\pi|\mathbf{x}|} \frac{\left(1 - \dfrac{M(x-x_0)}{|\mathbf{x}|}\right)\sigma}{\left[\dfrac{z^2}{|\mathbf{x}|^2}\left(1 - \dfrac{M(x-x_0)}{|\mathbf{x}|}\right)^2 + \sigma^2\right]}.$$

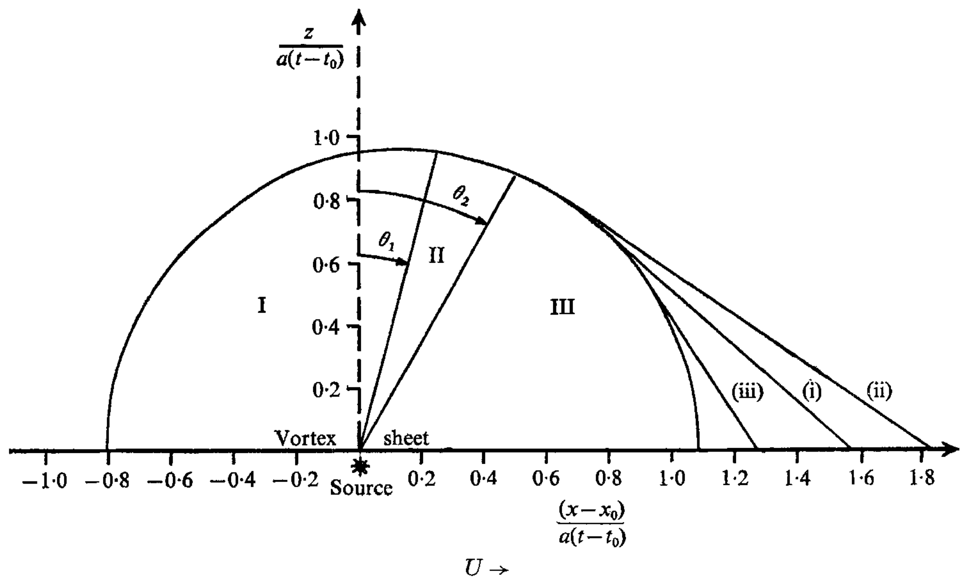

FIGURE 2. Section of the stable transmitted field of the acoustic pulse in the plane $y = y_0$. The Mach number $M = 3$ and the dimensionless distance of the source from the interface, $|z_0|/a(t-t_0)$, is equal to 0·05. Regions I, II and III are marked, and the curve shows the shape of the wave front. In regions I and III the acoustic field is that of a pure pulse, but in the zone of relative silence, region II, the pulse is diffuse and decays in intensity inversely as the distance from the wave front. Sections of the Mach cones are also shown: (i) the Mach cone of ϕ_{t_1}; (ii), (iii) the two Mach cones of ϕ_{t_2}. The angles θ_1 and θ_2 are given by $\sin\theta_1 = 1/(M+1)$, $\sin\theta_2 = 1/(M-1)$, in accordance with geometrical acoustics, so that region III exists only for $M > 2$.

This should be compared with the corresponding term in the solution of the *two-dimensional* reflexion problem treated by Friedland & Pierce. They derive an expression for the refracted arrival wave which propagates *ahead* of their reflected acoustic pulse. However, a close examination of their result reveals that, except where it merges with the reflected pulse, the refracted arrival wave falls off inversely as the distance from the source, and so is not responsible for the radiation of energy into the far field. In two dimensions only waves falling off inversely as the square root of the distance, or slower, would account for such radiation. Hence it is not at all surprising that our asymptotic analysis has not resulted in such a refracted arrival forerunner to the acoustic pulse.

4. The resonance modes

The geometrical acoustics solution presented in the previous section is valid at all points of the far field except those lying along certain critical directions in region III. Waves which propagate along these directions are in resonance with neutral oscillations which can be sustained by the interface, and have phase velocities in the x, y plane equal to the phase velocities of these neutral modes. In these directions the transmission coefficient $T(\mathbf{x})$ is infinite, which corresponds mathematically to the coincidence of the point of stationary phase in the ω plane and a real pole of the transmission coefficient. Actually there is now *no* contribution from stationary phase, a careful analysis showing that the stationary phase term tends continuously to zero as the resonance angle is approached. The excitation of these resonant modes must now be considered.

The denominator of the integrand of (3.1) may be factorized in the following manner

$$\omega^2\gamma_2+(\omega-Ul)^2\gamma_1 = a^2(\gamma_1+\gamma_2)(\gamma_1\gamma_2+l^2+m^2). \quad (4.1)$$

The first bracket on the right vanishes at $\omega = \frac{1}{2}Ul$ on the real axis provided that

$$M > 2\left[\frac{l^2+m^2}{l^2}\right]^{\frac{1}{2}}, \quad (4.2)$$

so that M must be at least greater than 2 for there to be any contribution from this pole. Miles (1958) has shown further that if

$$M > 2\sqrt{2}\left[\frac{l^2+m^2}{l^2}\right]^{\frac{1}{2}} \quad (4.3)$$

two real zeros of the second bracket on the right of (4.1) exist at

$$\omega = \frac{Ul}{2} \pm \frac{1}{2}[U^2l^2+4a^2(l^2+m^2)-4a(l^2+m^2)^{\frac{1}{2}}[U^2l^2+a^2(l^2+m^2)]^{\frac{1}{2}}]^{\frac{1}{2}}. \quad (4.4)$$

When the inequality (4.3) is not satisfied these two poles lie at conjugate points in the complex ω plane. This case, which gives rise to exponentially growing modes, will be considered later.

The contribution from the pole at $\omega = \frac{1}{2}Ul$ which lies outside the integration contour in the ω plane, ϕ_{t_1}, say, may be written immediately in the form

$$\phi_{t_1} = \frac{-1}{4\pi^2a^2}\iint_{-\infty}^{\infty} dl\,dm\,\frac{U|l|\,H([\frac{1}{4}M^2-1]\,l^2-m^2)\,|(\frac{1}{4}M^2-1)\,l^2-m^2|^{\frac{1}{2}}}{8m^2+(8-M^2)\,l^2}$$
$$+\exp[i\{l(x-x_0)+m(y-y_0)+\gamma_1(z+z_0)-\tfrac{1}{2}Ul\,(t-t_0)\}]. \quad (4.5)$$

It is convenient now to make the transformation $m = \mu|l|$, in which case (4.5) may be expressed in the form

$$\phi_{t_1} = \frac{-1}{4\pi^2a^2}\iint_{-\infty}^{\infty}\frac{dl\,d\mu\,U|l|\lambda H(\lambda)\exp[il\{(x-x_0)+\mu(y-y_0)+\lambda(z+z_0)-\frac{1}{2}U(t-t_0)\}]}{8\mu^2+(8-M^2)}, \quad (4.6)$$

where $\lambda = +[\frac{1}{4}M^2-1-\mu^2]^{\frac{1}{2}}.$

The method of stationary phase may now be applied to the μ integration, after which the integration with respect to l may be performed exactly (see Lighthill 1958). The analysis is straightforward and yields

$$\phi_{t_1} = \frac{M\left(\tfrac{1}{4}M^2-1\right)^{\frac{3}{4}}(z+z_0)^2 H(M-2)\, H\{\tfrac{1}{2}U(t-t_0) - (x-x_0) - \left(\tfrac{1}{4}M^2-1\right)^{\frac{1}{2}}[(y-y_0)^2+(z+z_0)^2]^{\frac{1}{2}}\}}{[M^2(y-y_0)^2+(8-M^2)(z+z_0)^2]\,\left|\tfrac{1}{2}U(t-t_0)-(x-x_0) - \left(\tfrac{1}{4}M^2-1\right)^{\frac{1}{2}}[(y-y_0)^2+(z+z_0)^2]^{\frac{1}{2}}\right|^{\frac{3}{2}}} \times \frac{1}{4\pi a[(y-y_0)^2+(z+z_0)^2]^{\frac{1}{4}}}. \quad (4.7)$$

This result may be interpreted as follows.

The surface

$$\tfrac{1}{2}U(t-t_0)-(x-x_0)-\left(\tfrac{1}{4}M^2-1\right)^{\frac{1}{2}}[(y-y_0)^2+(z+z_0)^2]^{\frac{1}{2}} = 0 \quad (4.8)$$

is a circular semi-cone whose vertex is at the point

$$x = x_0+\tfrac{1}{2}U(t-t_0), \quad y = y_0, \quad z = -z_0, \quad (4.9)$$

and whose axis lies on the line defined parametrically through t by (4.9). The surface is confined to the upstream side of the vertex and represents a Mach cone sweeping downstream at half the jet velocity, and inclined at each of its points at such an angle that it advances into the undisturbed fluid with a normal velocity equal to the velocity of sound. The cone may be extended forward through its vertex to intersect the surface $z = 0$ in a hyperbola. This hyperbola marks out a resonant disturbance which moves downstream without attenuation at speed $\frac{1}{2}U$, and whose motion has been triggered-off by the arrival at the interface of acoustic waves satisfying $\omega/l = \frac{1}{2}U$. The first such resonant waves to reach the surface fill up the region about the vertex of the hyperbola, and have the larger z components of wave-number. Resonant waves with smaller values of this wave-number component arrive later and gradually extend the disturbed region of the surface along both arms of the moving hyperbola. It is not difficult to see why such a Mach cone should develop. The arrival of a resonant wave at the interface excites the surface into this resonant mode of oscillation, which, once started does not need to be maintained by an external stimulus. Thus the surface behaves as if it were continuously being excited by the arrival of resonant modes and so continues to emit energy in the resonant direction. In fact this energy must be extracted directly from the shear layer.

Hence the correct interpretation of the solution (4.7) is that it represents the Mach cone of a neutrally stable disturbance moving supersonically downstream on the surface of the jet. Convergence of the Mach lines from each point of this disturbance through the vertex of the cone account for the singular behaviour of the solution at that point. This conclusion is borne out by a simple ray analysis which reveals that waves emitted from the source at time t_0 and satisfying $\omega/l = \frac{1}{2}U$ are refracted at the vortex sheet and ultimately come to lie on the cone (4.8) at time t.

Now these resonant waves are precisely those for which the geometrical acoustics approximation of the previous section fails. But since the normal velocity of the Mach cone is equal to the velocity of sound, it is clear that the acoustic pulse of (3.6) must in reality merge into the Mach cone solution (4.7) so that the latter may be imagined to touch the surface of the pulse. Also since this will occur at those points on the cone generated by resonant waves refracted at the surface, the curve of intersection of the pulse and cone marks the limit of the upstream extent of the cone, since the latter is non-existent before the arrival of the resonant waves. The mathematical reason for this cut-off is that for points further upstream the pole at $\omega/l = \frac{1}{2}U$ lies within the integration contour in the ω plane and so does not contribute to the solution. The cone otherwise lies downstream of the pulse and is more intense in that (*a*) its amplitude falls off inversely only as the square root of the distance from the jet, and (*b*) the three halves singularity of the solution (4.7) is stronger than the δ function of the pulse.

When condition (4.3) is satisfied there is a further resonant contribution to the solution ϕ_{t_2}, say, corresponding to the poles of the transmission coefficient at the points determined by (4.4). In view of the complexity of the expression for these points it is not possible to repeat in full the above analysis. It is possible, however, to make some general comments on the nature of the corresponding terms in the solution.

Consider the general expression for the solution given by (3.1); then the contribution due to a pole at a point given by (4.4) lying outside the integration contour in the ω plane is

$$\phi_{t_2} = \frac{1}{4\pi^2a^2}\iint_{-\infty}^{\infty} \frac{dl\,dm\,\omega(\omega-Ul)}{(\partial/\partial\omega)\,[\omega^2\gamma_2+(\omega-Ul)^2\gamma_1]}\,H\left(M-2\sqrt{2}\left(\frac{l^2+m^2}{l^2}\right)^{\frac{1}{2}}\right) \times \exp\left[i\{l(x-x_0)+m(y-y_0)+\gamma_1 z-\gamma_2 z_0-\omega(t-t_0)\}\right], \quad (4.10)$$

where γ_1, γ_2 are easily shown to be real. Consider the term

$$D = \frac{\omega(\omega-Ul)}{\partial\{\omega^2\gamma_2+(\omega-Ul)^2\gamma_1\}/\partial\omega}. \quad (4.11)$$

It is readily verified that D is a function of $(m/l)^2$ alone. If, as before, we make the substitution $m = \mu|l|$, and define

$$\overline{\gamma}_1 = \gamma_1/l, \quad \overline{\gamma}_2 = \gamma_2/l, \quad \Omega = \omega/l,$$

where now $\overline{\gamma}_1$, $\overline{\gamma}_2$, Ω are functions of μ^2 alone, then (4.10) takes the form

$$\phi_{t_2} = \frac{1}{4\pi^2a^2}\iint_{-\infty}^{\infty} dl\,d\mu\,|l|D(\mu^2)H(M-2\sqrt{2}\,(1+\mu^2)^{\frac{1}{2}}) \times \exp\left[il\{(x-x_0)+\mu(y-y_0)+\overline{\gamma}_1 z-\overline{\gamma}_2 z_0-\Omega(t-t_0)\}\right]. \quad (4.12)$$

Now apply stationary phase with respect to the μ integration. To do this we must solve

$$y-y_0+(\partial/\partial\mu)\{\overline{\gamma}_1 z-\overline{\gamma}_2 z_0-\Omega(t-t_0)\} = 0$$

for the stationary value, μ_0, say, after which the final result can be shown to have the general form

$$\phi_{t_2} = \frac{CH(M-2\sqrt{2})\,H\{\Omega(t-t_0)-(x-x_0)-\mu_0(y-y_0)-\overline{\gamma}_1 z+\overline{\gamma}_2 z_0\}}{4\pi a\sqrt{X}\,|\Omega(t-t_0)-(x-x_0)-\mu_0(y-y_0)-\overline{\gamma}_1 z+\overline{\gamma}_2 z_0|^{\frac{3}{2}}}, \quad (4.13)$$

where C is a dimensionless function of $(\mathbf{x}, t, M)$, and X has the dimensions of length and is proportional to the distance of $\mathbf{x}$ from the jet interface. The expression (4.13) has the same form as ϕ_{t_1} (equation (4.7)), and may be interpreted in a similar way. The surface

$$\Omega(t-t_0)-(x-x_0)-\mu_0(y-y_0)-\overline{\gamma}_1 z+\overline{\gamma}_2 z_0 = 0 \tag{4.14}$$

represents a Mach 'cone' obtained by eliminating μ from the equations

$$\Omega(t-t_0)-(x-x_0)-\mu(y-y_0)-\overline{\gamma}_1 z+\overline{\gamma}_2 z_0 = 0, \tag{4.15}$$

$$y-y_0+\frac{\partial}{\partial\mu}\{\overline{\gamma}_1 z-\overline{\gamma}_2 z_0-\Omega(t-t_0)\} = 0. \tag{4.16}$$

The two planes (4.15), (4.16) intersect along a Mach line which touches the Mach cone at each of its points in $z > 0$. Further, it may be shown that the phase surface (4.15) is tangential to the cone along such a line, as expected on physical grounds. The resonant disturbance producing such a Mach line lies at the intersection of the Mach line with the plane $z = 0$, i.e. at

$$\left.\begin{aligned} x &= x_0-z_0\mu(\partial\overline{\gamma}_2/\partial\mu)+\overline{\gamma}_2 z_0+\Omega(t-t_0),\\ y &= y_0+z_0(\partial\overline{\gamma}_2/\partial\mu)+(\partial\Omega/\partial\mu)\,(t-t_0),\\ z &= 0, \end{aligned}\right\} \tag{4.17}$$

and so moves over the interface at a velocity $\mathbf{v}$ given by

$$\mathbf{v} = \{\Omega, (\partial\Omega/\partial\mu), 0\},$$

which, in general, is different for different values of μ. Hence, since μ varies with the direction of propagation of the incident resonant wave, it is clear that the resonant energy sources located on the surface of the jet do not propagate at the same velocity, so that, unlike the hyperbolic disturbance considered previously, the *shape* of the resonant disturbance now *varies with time*, producing continuous modification of the Mach cone.

Note further that for each value of μ equation (4.4) gives two resonant modes, hence ϕ_{t_2} involves *two* Mach cones. Each must touch the acoustic pulse propagating into region III in much the same manner as previously described. Again the cones do not extend upstream beyond their curves of contact with the acoustic pulse, as before such a situation would involve poles lying within the integration contour in the ω plane.

We are now in a position to collect together our results for the stable field of the transmitted wave. This is illustrated in figure 2, which shows a section in the plane $y = y_0$ which contains the source.

5. The unstable modes

It remains to consider the contribution to the transmitted wave from those poles of the transmission coefficient which lie at complex values of ω. These occur at the points

$$\omega = \tfrac{1}{2}Ul \pm \tfrac{1}{2}i\,|U^2l^2+4a^2(l^2+m^2)-4a(l^2+m^2)^{\frac{1}{2}}\,(U^2l^2+a^2(l^2+m^2))^{\frac{1}{2}}|^{\frac{1}{2}}, \tag{5.1}$$

provided that
$$M < 2\sqrt{2}\left(\frac{l^2+m^2}{l^2}\right)^{\frac{1}{2}}. \tag{5.2}$$

In any far field analysis poles which lie in the lower half plane will give exponentially small contributions and may therefore be neglected. Hence we shall consider only the effect of those poles which lie above the real ω axis and outside the integration contour in the ω plane.

In view of the nature of the expression (5.1) it is not possible to carry out an exact analysis, even using steepest descents. However, since we have growing modes, those modes with the largest growth rate will eventually dominate the solution (see also Benjamin 1961), and it seems desirable, therefore, that any approximate scheme of evaluation should tend to concentrate on these modes. For each fixed wave-number component l, the most rapidly growing waves satisfy $|m/l| \gg 1$. If we suppose, in fact that $|m| \gg |l|\,(M^2-1)^{\frac{1}{2}}$, then (5.1) may be approximated by
$$\omega = \tfrac{1}{2}Ul + \tfrac{1}{2}iU|l|. \tag{5.3}$$

A little analysis reveals that the condition for the pole at (5.3) to lie outside the integration contour, and so contribute to the transmitted wave, is
$$z - z_0 < \tfrac{1}{2}a(t-t_0), \tag{5.4}$$
which limit should therefore mark the depth of penetration of the dominant mode into the stationary ambient fluid.

In keeping with the above approximation we may also set
$$\gamma_1 = \gamma_2 = i(l^2+m^2)^{\frac{1}{2}}. \tag{5.5}$$

Now the growth rate of an infinitely small disturbance is infinitely large. This means that after integrating around the complex pole at (5.3), the subsequent wave-number integration will diverge. In practice, however, there will be an upper cut-off to the wave-number spectrum of the initial disturbance. Thus in order to gain further insight into the nature of the growing instability we shall suppose this to occur at $|l| = R$. If further the range of integration over m is restricted to $|m| > |l|\,(M^2-1)^{\frac{1}{2}}$ in accordance with our previous approximation, then by expanding about the points of maximum growth rate in (l, m) space, we readily derive the following approximate expression for ϕ_{t_u}, the unstable contribution to the transmitted wave:
$$\phi_{t_u} = \frac{C}{4\pi a^2}\cos\{R[(x-x_0) \pm (M^2-1)^{\frac{1}{2}}(y-y_0) - \tfrac{1}{2}U(t-t_0)] + \epsilon\} \times \frac{\exp[R\{\tfrac{1}{2}U(t-t_0) - M(z-z_0)\}]}{(z-z_0)[(z-z_0)M - \tfrac{1}{2}U(t-t_0)]}, \tag{5.6}$$
where C and ϵ are functions of M and R alone. In view of the nature of this approximation, which assumes the existence of a non-zero maximum growth rate given by the argument of the exponential in (5.6), this solution is clearly not valid in the neighbourhood of
$$z - z_0 - \tfrac{1}{2}a(t-t_0) = 0.$$

We have already seen, moreover, that there is in fact no contribution from these dominant modes if $z - z_0 > \tfrac{1}{2}a(t-t_0)$.

The expression for ϕ_{t_u} is interpreted as representing two growing disturbances moving outwards obliquely to the jet flow along the lines

$$y - y_0 = \pm (x - x_0)(M^2 - 1)^{\frac{1}{2}} \tag{5.7}$$

at a velocity equal to $UM/2(M^2-1)^{\frac{1}{2}}$. The region between these two disturbances contains other growing modes, but with smaller growth rates. Hence we have the picture of an initial disturbance triggering-off an instability inside a wedge of angle $2\tan^{-1}((M^2-1)^{\frac{1}{2}})$ downstream of the source on the surface of the jet, the maximum growth rates being along the edges of the wedge. Practical estimates of the growth rates would depend on a knowledge of the wave-number cut-off R, which should typically be of the order of the inverse shear layer thickness.

6. Conclusion

We may summarise the general features of the acoustic field of the pulse as follows:

The arrival of the pulse at the vortex sheet always triggers-off the Kelvin–Helmholtz instability of the interface. This results in the generation of a growing disturbance which moves downstream at half the jet velocity, its influence being largely confined to the neighbourhood of the interface. The effect is independent of the Mach number of the jet flow.

Except in the zone of relative silence, the transmitted pulse may be predicted from a classical ray theory analysis, the amplitude being given in terms of a transmission coefficient. The pulse is diffuse in the zone of relative silence, and switches on to a finite level at the wave front, the amplitude then decaying inversely as the distance behind the front.

When the Mach number exceeds 2 the arrival of the pulse triggers-off a neutral disturbance in the interface which proceeds to move downstream at half the jet velocity generating a Mach cone running ahead of the acoustic pulse. Two similar disturbances are created if the Mach number is greater than $2\sqrt{2}$. These Mach waves greatly augment the acoustic pulse strength at their curves of contact with the pulse, and correspond to infinities of the transmission coefficient.

Finally let us note that it is precisely in the region downstream of the source where ideas of geometrical acoustics fail. Our analysis predicts the growth of instabilities together with the propagation of broad fronted pulses of enhanced amplitude into this region ahead of the acoustic pulse. It is evident, therefore, that an explanation of the experimentally observed zone of silence downstream of the source must await a more complete and non-linear analysis.

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REFERENCES

Benjamin, T. B. 1961 *J. Fluid Mech.* **10**, 401.

Berman, C. H. & Ffowcs Williams, J. E. 1970 *J. Fluid Mech.* **42**, 151.

Franken, P. & Ingard, U. 1956 *J. Acoust. Soc. Am.* **28**, 126.

Friedland, A. B. & Pierce, A. D. 1969 *Phys. Fluids*, **12**, 1148.

Gottlieb, P. 1960 *J. Acoust. Soc. Am.* **32**, 1117.

Hatanaka, H. 1949 *J. Soc. Sci. Culture* **2**, 3; cited by *Appl. Mech. Rev.* **2**, 897.

Keller, J. B. 1955 *J. Acoust. Soc. Am.* **27**, 1044.

Landau, L. 1944 *C.R. Acad. Sci. U.S.S.R.* **44**, 139.

Lighthill, M. J. 1952 *Proc. Roy. Soc.* A **211**, 564.

Lighthill, M. J. 1958 *Introduction to Fourier Analysis and Generalized Functions.* Cambridge University Press.

Lighthill, M. J. 1960 *Phil. Trans. Roy. Soc.* A **252**, 397.

Miles, J. W. 1957 *J. Acoust. Soc. Am.* **29**, 226.

Miles, J. W. 1958 *J. Fluid Mech.* **4**, 538.

Miles, J. W. & Fejer, J. A. 1963 *J. Fluid Mech.* **15**, 335.

Morse, P. M. & Ingard, K. U. 1968 *Theoretical Acoustics.* New York: McGraw-Hill.

Pai, S. I. 1954 *J. Aero Sci.* **21**, 325.

Rayleigh, Lord 1945 *The Theory of Sound*, Vol. 2. New York: Dover.

Ribner, H. S. 1957 *J. Acoust. Soc. Am.* **29**, 435.

Ribner, H. S. 1964 In *Advances in Applied Mathematics.* (Ed. H. L. Dryden & T. H. Von Karman.) New York: Academic.

Rudnick, I. 1946 *J. Acoust. Soc. Am.* **17**, 245.